

Complex numbers and symmetries in quantum mechanics, and a nonlinear superposition principle for Wigner functions

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Abstract

Complex numbers appear in the Hilbert space formulation of quantum mechanics, but not in the formulation in phase space. Quantum symmetries are described by complex, unitary or antiunitary operators defining ray representations in Hilbert space, whereas in phase space they are described by real, true representations. Equivalence of the formulations requires that the former representations can be obtained from the latter and *vice versa*. Examples are given. Equivalence of the two formulations also requires that complex superpositions of state vectors can be described in the phase space formulation, and it is shown that this leads to a nonlinear superposition principle for orthogonal, pure-state Wigner functions. It is concluded that the use of complex numbers in quantum mechanics can be regarded as a computational device to simplify calculations, as in all other applications of mathematics to physical phenomena.

Key words: Wigner functions, complex quantum mechanics, quantum symmetries, nonlinear superposition principle, quantum mechanics in phase space

1 Introduction

One of the remarkable features of quantum mechanics as usually formulated is the fundamental role played there by the complex numbers. This appears to distinguish quantum mechanics from other mathematical models of natural phenomena. Theoreticians use complex numbers routinely in most applications of mathematics, starting with the solution of simple quadratic equations. However, in all other models, the complex numbers are introduced only as a computational tool. Not only are the ‘observables’ of these models real but, invariably, so also are their defining equations.

Feynman and Dirac were prominent among those who have stressed the crucial role of complex probability amplitudes in quantum mechanics.

To quote Feynman: *It has been found that all processes so far observed can be understood in terms of the following prescription: To every process there corresponds an amplitude (a complex number); with proper normalization the probability of the process is equal to the absolute square of the amplitude [1].*

And Dirac: *So if one asks what is the main feature of quantum mechanics, I feel inclined now to say that it is ... the existence of probability amplitudes which underly all atomic processes. Now a probability amplitude is related to experiment but only partially. The square of its modulus is something that we can observe. That is the probability which the experimental people get. But besides that there is a phase ... (which) is all important because it is the source of all interference phenomena but its physical meaning is obscure [2].*

The essential place of complex numbers in quantum mechanics as usually formulated is evident in Schrödinger’s time-dependent wave equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t), \quad (1)$$

where $\psi(t)$ is the state vector of a quantum system and \hat{H} is its Hamiltonian operator. For spinless, nonrelativistic systems, \hat{H} is a real operator in the usual coordinate realization of Hilbert space, and the radical departure that the complex equation (1) then represents, from the reality of the governing

equations of all earlier models of natural phenomena, was a departure that Schrödinger made uneasily [3].

There is a formulation of quantum mechanics [4], [5], [6] – I will refer to it as the phase-space formulation – that has become popular as an arena in which to investigate such matters as quantum chaos [7], quantum tomography [8], [9], the relationship between quantum and classical mechanics [10], [11], and the nature of quantization [12], [13]. What I would like to draw attention to here is that the phase-space formulation is purely real – there are no complex numbers to be seen in the defining equations. Despite this, it has been stated [12], [14] that the phase-space formulation is equivalent to the more familiar formulation in terms of hermitian operators acting on a complex Hilbert space. In view of the remarks above, one may well ask how this can be so.

In what follows, I shall discuss two aspects of this surprising state of affairs. The first concerns representations of symmetry groups and algebras. In the Hilbert space formulation, elements of a symmetry group or more generally, of any group of automorphisms of the set of quantum states, are represented by unitary or antiunitary operators that define ray representations of the group as a whole, in accordance with Wigner’s celebrated theorem [15, 16, 17]. In contrast, in the phase-space formulation, elements of a symmetry group are represented by real unitary operators that define a true representation. If the phase-space formulation is equivalent to the one in Hilbert space, then the complex ray representations must be recoverable from the real true ones, and *vice versa*. That this is indeed the case [18] is remarkable from the point of view of the mathematics and the physics involved.

The second aspect concerns the fact that in the phase-space formulation, quantum states are described by real Wigner functions, whereas in Hilbert space, one has state vectors that can be superposed, and that have complex phases. We shall see that there is a corresponding nonlinear superposition principle for pure-state Wigner functions, that incorporates the relative phase of state vectors that are supersposed.

2 Group representations

Two examples for systems with one linear degree of freedom illustrate the strikingly different treatment of group representations in the Hilbert space

and phase space formulations [18].

Consider firstly the Heisenberg-Weyl group, which in the abstract may be considered as the 2-parameter abelian Lie group of transformations $g(a, b)$, $a, b \in \mathbb{R}$, acting on real, square-integrable functions $F(q, p)$ of two real variables as

$$g(a, b) : F(q, p) \longrightarrow F(q + a, p - b), \quad (2)$$

We can write, on suitably smooth F ,

$$g(a, b) = \exp[ax_1 + bx_2], \quad g(a, b)g(c, d) = g(a + c, b + d).$$

$$x_1 = \partial/\partial q, \quad x_2 = -\partial/\partial p, \quad [x_1, x_2] = 0. \quad (3)$$

and we note the appearance of the 2-dimensional real Lie algebra spanned by x_1 and x_2 .

The familiar unitary ray representation of this group on Hilbert space has

$$g(a, b) \longrightarrow U(a, b),$$

$$U(a, b)U(c, d) = e^{i(ad-bc)/2\hbar} U(a + c, b + d). \quad (4)$$

With

$$U(a, b) = \exp[i(a\hat{P} + b\hat{Q})/\hbar], \quad (5)$$

we have

$$[\hat{Q}, \hat{P}] = i\hbar\hat{I} \quad (6)$$

where \hat{I} is the unit operator, so that at the level of the Lie algebra the ray representation of the group is associated with a central extension, characterised by Planck's constant \hbar . In either the coordinate realization or the momentum realization of Hilbert space, one of the operators \hat{Q} , \hat{P} is real, the other pure imaginary, and again we see the essential role played by the complex numbers.

Because this ray representation and central extension lie at the very heart of the Hilbert space formulation, it is all the more surprising to find that in the phase space formulation, the Heisenberg-Weyl group simply has its defining representation (2, 3). This representation is real and true; not only the complex numbers, but also Planck's constant, are nowhere to be seen.

The second example is provided by the time reversal group of order 2, with elements g , e acting on functions $F(q, p)$ as

$$g : F(q, p) \longrightarrow F(q, -p), \quad e : F(q, p) \longrightarrow F(q, p), \quad (\Rightarrow g^2 = e). \quad (7)$$

In the Hilbert space formulation, e is represented by the unit operator $\hat{U}_e = \hat{I}$, and g by an antiunitary operator \hat{A}_g . In the coordinate realization, for example, we may have

$$\hat{A}_g : \psi(x) \longrightarrow \psi^*(x), \quad \hat{U}_e : \psi(x) \longrightarrow \psi(x), \quad (\Rightarrow A_g^2 = U_e). \quad (8)$$

Once again, the representation in the phase space formulation is simply the defining representation (7), and in sharp contrast to the Hilbert space representation, is real and unitary.

It is not hard to find the source of these differences. The real structure that is inherent in the phase space formulation is in fact already present in the Hilbert space formulation. To see this, we recall [6], [18] that the mapping from the Hilbert space formulation to the phase space formulation is the Weyl-Wigner transform \mathcal{W} , whose inverse \mathcal{W}^{-1} is Weyl's quantization map. The transform \mathcal{W} does not act on state vectors in Hilbert space at all. Rather it maps the real Hilbert space \mathcal{T} of hermitian Hilbert-Schmidt operators \hat{A} that act on the space of state vectors, onto the real Hilbert space \mathcal{K} of square integrable functions A on phase space. Conversely, \mathcal{W}^{-1} carries such functions A back into hermitian operators \hat{A} (that is, it quantizes them). If $A_K(x, y)$ is the kernel of \hat{A} , regarded as an integral operator in the coordinate realization of Hilbert space, then the action of \mathcal{W} and \mathcal{W}^{-1} is defined by

$$\begin{aligned} \mathcal{W}(\hat{A})(q, p) &= \int A_K(q - y/2, q + y/2) e^{ipy/\hbar} dy, \\ \mathcal{W}^{-1}(A)_K(x, y) &= \frac{1}{2\pi\hbar} \int A([x + y]/2, p) e^{ip(x-y)/\hbar} dp. \end{aligned} \quad (9)$$

With a suitably generalized interpretation of the integrals, these formulas also define the action of \mathcal{W} on hermitian operators that lie outside \mathcal{T} , such as \hat{Q} , \hat{P} , and the action of \mathcal{W}^{-1} on real-valued phase space functions that lie outside \mathcal{K} , such as q and p .

It can be seen from this that the question of the equivalence of the Hilbert space and phase space formulations of quantum mechanics is really a question

as to whether or not, in the usual Hilbert space formulation, one needs state vectors to truly describe states, or whether it is sufficient to work only with density operators and the real algebra of hermitian operators.

The density operator $\hat{\rho}$ for the state of the system involved, lies in \mathcal{T} and is mapped by \mathcal{W} into $(2\pi\hbar \times)$ Wigner's pseudo probability density function $W_\rho \in \mathcal{K}$ on phase space. But there is no mapping of the complex state vector, which has no image appearing in the phase space formulation.

A ray representation $\hat{\pi}$ of a group G , defined by unitary and antiunitary operators acting on state vectors ψ , also defines a real, true representation $\Pi_{\mathcal{T}}$ of G on \mathcal{T} ,

$$g \in G : \psi \longrightarrow \hat{\pi}(g)\psi,$$

$$\hat{A} \longrightarrow \Pi_{\mathcal{T}}(g)(\hat{A}) = \hat{\pi}(g)\hat{A}\hat{\pi}(g)^\dagger. \quad (10)$$

It is this real, true representation $\Pi_{\mathcal{T}}$ that is mapped by the Weyl-Wigner transform onto a real, true representation $\Pi_{\mathcal{K}}$ on square-integrable functions on phase space.

From (10) it can be seen that $\Pi_{\mathcal{K}}$ is isomorphic to the tensor product of the ray representation $\hat{\pi}$ with its contragredient $\hat{\pi}^C$; elsewhere [18] we have called $\Pi_{\mathcal{K}}$ the Weyl-Wigner product of $\hat{\pi}$ and $\hat{\pi}^C$.

This explains why groups have real, true representations in the phase-space formulation. But if that formulation is indeed equivalent to the Hilbert space formulation, it must be possible to recover the underlying ray representations from the phase-space ones. In effect, we must be able to 'factorize' $\Pi_{\mathcal{K}}$, or $\Pi_{\mathcal{T}}$, and determine $\hat{\pi}$. That this is possible is surprising, especially when it is recalled that at the level of the Lie algebra we have to determine a centrally extended representation, with associated parameter(s), from a true one. Because the representation $\Pi_{\mathcal{K}}$ can typically be regarded as the defining representation of the group of interest, as in the examples of the Heisenberg-Weyl group and time reversal group above, it can be said that in finding $\hat{\pi}$ we are expressing Wigner's Theorem in a constructive way for that group.

It has to be emphasized that not every real, true representation on \mathcal{K} of a given group can be factorized in this way. Necessary and sufficient conditions for this to be possible have been determined elsewhere [18]. We shall expand here on one aspect only: the construction of $\hat{\pi}$ in the case of a 1-parameter Lie group, whose representation $\Pi_{\mathcal{K}}$ is generated by a real, skew-adjoint operator

on \mathcal{K} , which we may think of as an integral operator $\hat{\alpha}$ with real kernel $\alpha_{\mathcal{K}}$,

$$(\hat{\alpha}F)(q, p) = \int \alpha_{\mathcal{K}}(q, p, q', p') F(q', p') dq' dp'. \quad (11)$$

In this case, the necessary and sufficient condition on $\hat{\alpha}$ for factorizability is [18]

$$\begin{aligned} & \int \sin[(qp_1 + pq_1)/\hbar] R(q_1, p_1, q', p') dq_1 dp_1 \\ &= \int \sin[(q_1 p + q_1 p' + p_1 q + p_1 q')/2\hbar] R(q_1, p_1, [q + q']/2, [p + p']/2) dq_1 dp_1 \\ & - \int \sin[(q_1 p - q_1 p' + p_1 q' - p_1 q)/2\hbar] R(q_1, p_1, [q' - q]/2, [p' - p]/2) dq_1 dp_1, \end{aligned} \quad (12)$$

where

$$R(q, p, q', p') = \alpha_{\mathcal{K}}([q' - q]/2, [p' + p]/2, [q' + q]/2, [p' - p]/2). \quad (13)$$

When this condition holds, we can obtain the (complex) hermitian generator \hat{A} of the representation $\hat{\pi}$ on Hilbert space by setting

$$A(q, p) = c + \int \sin[(qp' + pq')/\hbar] R(q', p', q, p) dq' dp', \quad (14)$$

where c is an arbitrary constant, and then

$$\hat{A} = \mathcal{W}^{-1}(A). \quad (15)$$

Note that \hat{A} is defined only up to the addition of a multiple of \hat{I} by c , leading to an arbitrary overall phase in the representation of the 1-parameter group on Hilbert space.

It is surprising that complex ray representations can be obtained from true representations by this factorization process, which involves the ‘quantization’ step (15). At the level of the corresponding Lie algebra, we may obtain a representation of a central extension of that algebra in this way. Clearly any extension parameters must be built into the transforms (12) and \mathcal{W}^{-1} (namely \hbar), or else appear already in the true representation on phase space. Both possibilities arise in practice [18].

As an example, consider again the Heisenberg-Weyl group for one degree of freedom, with representation (2, 3) on phase space. In this case, as seen from (3), $\alpha_1 = x_1 = \partial/\partial q$, $\alpha_2 = x_2 = -\partial/\partial p$, leading from (11) to

$$\alpha_{1K}(q, p, q', p') = \delta'(q-q')\delta(p-p') , \quad \alpha_{2K}(q, p, q', p') = -\delta(q-q')\delta'(p-p') , \quad (16)$$

and hence from (14) and (15) to

$$A_1(q, p) = p + c_1 , \quad A_2(q, p) = q + c_2 ,$$

$$\hat{A}_1 = \hat{P} + c_1 \hat{I} , \quad \hat{A}_2 = \hat{Q} + c_2 \hat{I} . \quad (17)$$

The constant terms in \hat{A}_1 , \hat{A}_2 can be removed by a unitary transformation generated by \hat{Q} followed by a unitary transformation generated by \hat{P} , leaving \hat{P} and \hat{Q} satisfying (6).

3 Superposition of Wigner functions

According to the first of (9), the Wigner function W_ρ corresponding to $([2\pi\hbar]^{-1}\times)$ the pure-state density operator $\hat{\rho}$ with kernel $\rho_K(x, y) = \varphi(x)\varphi^*(y)$ is

$$W_\rho(q, p) = \frac{1}{2\pi\hbar} \int \varphi(q - y/2)\varphi^*(q + y/2) e^{ipy/\hbar} dy . \quad (18)$$

Here $\varphi(x)$ is the representative of a normalized pure state vector corresponding to $\hat{\rho}$, in the coordinate realization of Hilbert space. It is clear that $\varphi(x)$ can be recovered from $W_\rho(q, p)$ by applying the second of (9). We have [19]

$$\varphi(x)\varphi^*(x_0) = \int W_\rho([x + x_0]/2, p) e^{ip(x-x_0)/\hbar} dp , \quad (19)$$

which determines a fixed complex multiple of $\varphi(x)$ so long as we choose any particular x_0 such that $\varphi(x_0) \neq 0$ and hence such that the left-hand-side of (19) is not identically equal to zero. What we wish to show here is that a complex superposition of two state vectors, including their relative phase, can also be described in terms of the two corresponding real Wigner functions.

Suppose then that we are given two normalized, orthogonal pure-state wave functions, $\varphi_1(x)$, $\varphi_2(x)$, from which we have constructed the corresponding density operators $\hat{\rho}_1$, $\hat{\rho}_2$, and the corresponding Wigner functions

$W_1(q, p)$, $W_2(q, p)$. We consider the normalized superposition

$$\varphi(x) = a_1 \varphi_1(x) + a_2 \varphi_2(x), \quad |a_1|^2 + |a_2|^2 = 1. \quad (20)$$

Choosing particular values x_1 , x_2 such that $\varphi_1(x_1) \neq 0$, $\varphi_2(x_2) \neq 0$, we rewrite (20) as

$$\varphi(x) = c_1 \varphi_1(x) \varphi_1^*(x_1) + c_2 \varphi_2(x) \varphi_2^*(x_2), \quad |c_1|^2 |\varphi_1(x_1)|^2 + |c_2|^2 |\varphi_2(x_2)|^2 = 1. \quad (21)$$

The kernel of the density operator corresponding to the pure state $\varphi(x)$ is then

$$\begin{aligned} \rho_K(x, y) &= \varphi(x) \varphi^*(y) = |a_1|^2 \varphi_1(x) \varphi_1^*(y) + |a_2|^2 \varphi_2(x) \varphi_2^*(y) \\ &\quad + c_1 c_2^* \varphi_1(x) \varphi_1^*(x_1) \varphi_2^*(y) \varphi_2(x_2) + c_2 c_1^* \varphi_2(x) \varphi_2^*(x_2) \varphi_1^*(y) \varphi_1(x_1), \\ &= |a_1|^2 \rho_{1K}(x, y) + |a_2|^2 \rho_{2K}(x, y) \\ &\quad + c_1 c_2^* \rho_{1K}(x, x_1) \rho_{2K}^*(y, x_2) + c_2 c_1^* \rho_{2K}(x, x_2) \rho_{1K}^*(y, x_1). \end{aligned} \quad (22)$$

Applying the first of (9) to $\rho_K/2\pi\hbar$, we obtain

$$\begin{aligned} W_\rho(q, p) &= |a_1|^2 W_1(q, p) + |a_2|^2 W_2(q, p) \\ &\quad + \frac{c_1 c_2^*}{2\pi\hbar} \int \rho_{1K}(q - y/2, x_1) \rho_{2K}^*(q + y/2, x_2) e^{ipy/\hbar} dy \\ &\quad + \frac{c_2 c_1^*}{2\pi\hbar} \int \rho_{2K}(q - y/2, x_2) \rho_{1K}^*(q + y/2, x_1) e^{ipy/\hbar} dy. \end{aligned} \quad (23)$$

Next rewriting ρ_{1K} and ρ_{2K} using the second of (9), we see that the third term on the RHS of (23) can be written as

$$\begin{aligned} &\frac{c_1 c_2^*}{2\pi\hbar} \iiint W_1([q - y/2 + x_1]/2, p_1) W_2([q + y/2 + x_2]/2, p_2) \\ &\quad \times e^{ip_1[q - y/2 - x_1]/\hbar} e^{-ip_2[q + y/2 - x_2]/\hbar} e^{ipy/\hbar} dp_1 dp_2 dy, \end{aligned} \quad (24)$$

where we have noted the reality of W_2 . Because the fourth term on the RHS of (23) is the complex conjugate of the third, we see that the sum of the third and fourth terms can be written as

$$\begin{aligned} & \frac{|c_1||c_2|}{\pi\hbar} \iiint W_1([q - y/2 + x_1]/2, p_1) W_2([q + y/2 + x_2]/2, p_2) \\ & \times \cos[\epsilon + (2p - p_1 - p_2)y/2 + (p_1 - p_2)q - p_1x_1 + p_2x_2] dp_1 dp_2 dy, \end{aligned} \quad (25)$$

where we have written

$$c_1 c_2^* = |c_1||c_2|e^{i\epsilon}, \quad 0 \leq \epsilon < 2\pi. \quad (26)$$

Next we note that

$$|a_1|^2 = |c_1|^2 |\varphi_1(x_1)|^2 = |c_1|^2 \int W(x_1, p) dp, \quad (27)$$

with a similar relation for $|a_2|^2$. Then we can write (23) as

$$\begin{aligned} W(q, p) = & |c_1|^2 \left\{ \int W(x_1, p') dp' \right\} W_1(q, p) + |c_2|^2 \left\{ \int W(x_2, p') dp' \right\} W_2(q, p) \\ & + \frac{|c_1||c_2|}{\pi\hbar} \iiint W_1([q - y/2 + x_1]/2, p_1) W_2([q + y/2 + x_2]/2, p_2) \\ & \times \cos[\epsilon + (2p - p_1 - p_2)y/2 + (p_1 - p_2)q - p_1x_1 + p_2x_2] dp_1 dp_2 dy. \end{aligned} \quad (28)$$

This is a nonlinear superposition rule for Wigner functions, expressing a new Wigner function W in terms of two given ones, W_1 and W_2 , two nonnegative coefficients $|c_1|$, $|c_2|$, and a phase angle ϵ between 0 and 2π as in (26). The two coordinates x_1 , x_2 in (25) are arbitrary, except that we must have

$$\int W_1(x_1, p) dp \neq 0, \quad \int W_2(x_2, p) dp \neq 0, \quad (29)$$

for consistency with our assumption above that $\varphi(x_1) \neq 0$ and $\varphi(x_2) \neq 0$. Since we also assumed that φ_1 and φ_2 are orthogonal, we must also require that W_1 and W_2 are orthogonal, that is

$$\iint W_1(q, p) W_2(q, p) dq dp = \frac{1}{2\pi\hbar} \left| \int \varphi_1^*(x) \varphi_2(x) dx \right|^2 = 0, \quad (30)$$

and since we assumed that W_1 and W_2 correspond to pure states, they must also satisfy the pure-state conditions [19]

$$\int W_1(q, p)^2 dq dp = \frac{1}{2\pi\hbar}, \quad \int W_2(q, p)^2 dq dp = \frac{1}{2\pi\hbar}, \quad (31)$$

as well as the normalization conditions

$$\int W_1(q, p) dq dp = 1, \quad \int W_2(q, p) dq dp = 1. \quad (32)$$

It can now be seen that (20) and (28) are equivalent. If we are given two orthogonal, normalized wave functions φ_1 and φ_2 and form the superposition (20), then from φ we can construct W_ρ as in (28) following the steps outlined above. Conversely, if we start with W_1 and W_2 , we can find an x_1 and x_2 such that (29) hold, and form the nonlinear superposition (28) for some choice of $|c_1|$, $|c_2|$ and ϵ such that the second of (26) is satisfied. Note that fixing $|c_1|$, $|c_2|$ and ϵ fixes complex c_1 and c_2 , up to an overall phase, as for example in (36) below. We can also construct $\varphi_1(x)\varphi_1^*(x_1)$ and $\varphi_2(x)\varphi_2^*(x_2)$ from W_1 and W_2 , as in (19), and then recover φ as in (20), up to an overall phase.

The result can also be expressed as follows. Given two orthogonal pure-state Wigner functions W_1 and W_2 satisfying (30), (31) and (32), we choose any constants x_1 and x_2 such that (29) hold, and any constants A , B and ϵ satisfying

$$A \geq 0, \quad B \geq 0, \quad 0 \leq \epsilon < 2\pi, \quad (33)$$

and we form

$$\begin{aligned} T(q, p) = & A^2 \left\{ \int W(x_1, p') dp' \right\} W_1(q, p) + B^2 \left\{ \int W(x_2, p') dp' \right\} W_2(q, p) \\ & + \frac{AB}{\pi\hbar} \iiint W_1([q - y/2 + x_1]/2, p_1) W_2([q + y/2 + x_2]/2, p_2) \\ & \times \cos[\epsilon + (2p - p_1 - p_2)y/2 + (p_1 - p_2)q - p_1 x_1 + p_2 x_2] dp_1 dp_2 dy. \end{aligned} \quad (34)$$

Then

$$W(q, p) = T(q, p) / \int T(q', p') dq' dp' \quad (35)$$

is a normalized pure state Wigner function. It corresponds to the wave function superposition (21), where $\varphi_1(x)\varphi_1^*(x_1)$ and $\varphi_2(x)\varphi_2^*(x_2)$ are constructed

from W_1 and W_2 as in (19), and

$$|c_1| = A / \left\{ \int T(q, p) dq dp \right\}^{1/2}, \quad |c_2| = B / \left\{ \int T(q, p) dq dp \right\}^{1/2},$$

$$c_1 = |c_1|, \quad c_2 = |c_2| e^{-i\epsilon}. \quad (36)$$

4 Conclusions

Quantum symmetries and complex superpositions of pure, orthogonal quantum states, can both be described in the phase space formulation of quantum mechanics, without the use of complex numbers. The description of symmetries is much simpler in phase space, but the description of superpositions is much more complicated. The description of non-orthogonal superpositions is even more complicated, and has not been attempted here.

Note that the relative phase of the superposed wave functions, whose *physical meaning is obscure* [2], is essentially the ϵ appearing in the nonlinear superposition formulas (28) and (34).

We conclude that the phase space formulation does indeed appear capable of reproducing all aspects of quantum mechanics. In the case of the superposition of quantum states however, this is only be achieved at the cost of much greater complication.

If we wish to think of the phase space formulation as the more fundamental, arising directly from a deformation of classical mechanics in phase space [12], we can think of the formulation of quantum mechanics in Hilbert space, and the associated introduction of complex numbers, as a computational device to make calculations easier. From this point of view, the appearance of complex numbers in quantum mechanics is on a similar footing to their appearance in other applications of mathematics to natural phenomena.

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